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Equator projection sundials

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A discussion of elliptical analemmatic sundials shows that they are a subset of a group which has the property that the time is indicated by the intersection of a gnomon shadow with the projection of the equator circle.
A systematic geometrical treatment of these leads to the discovery and presentation of a hitherto unknown subset designated the 'central projection dials'.

Introduction

Several kinds of equator projection sundial have been known since the seventeenth century. The original inventor is unknown, but in 1640 De Vaulezard *(ref. 1)* published an account of the construction of a horizontal sundial in which the hour points were situated on the circumference of an ellipse and the vertical gnomon had to be moved, according to the date, along the short axis of the ellipse. In 1654, Samuel Foster *(ref. 2)* published a book in which he treated the same subject extensively. He also described varieties of sundials that had the hour points on a circle or on a straight line; and a combination of two sundials in which the hour points were situated on the same circle, but which had two differently directed gnomons. This combination has been rediscovered several times in the following centuries. Since Foster's work, no new kinds of equator projection sundial have been found, but the theory has been made very much simpler. In 1757 Jerome Lalande *(ref. 3)* could still write: "This problem [the proof of the exactness of the construction] is one of the most difficult of the whole gnomonics". Among those who kept themselves busy with the theory of the analemmatic sundials one finds many well known astronomers and mathematicians. *(ref. 4)* The best treatment was found in 1951, by P. Terpstra. *(ref. 5)* He treated the analemmatic sundial as a projection of the equator circle, the time being indicated by the intersection of a gnomon shadow with this projection. After further examination, building on Terpstra's proof, it appeared to me that all known kinds of equator projection sundial can be derived from one principle of construction. This possibility showed up an endless number of new varieties. Furthermore, I discovered a still unknown family of sundials, based on a closely related principle of construction *(ref. 11)* .

Two equator projection sundials

One of the oldest and best known sundials of this kind can be seen in the square of Brou Cathedral *(ref. 6)* (in Bourg-en-Bresse, Ain, France). The hour points have been hewn in a stone ellipse with a long axis of 10m (33 ft) and a short axis of 8m (26 ft). The short axis lies in the direction of the meridian and on this axis a 4m (13 ft) long date scale has been fixed. If one puts a vertical gnomon on the exact place on the scale corresponding to the date, the gnomon's shadow intersects the ellipse at a position corresponding to the local apparent time (Figure 1).

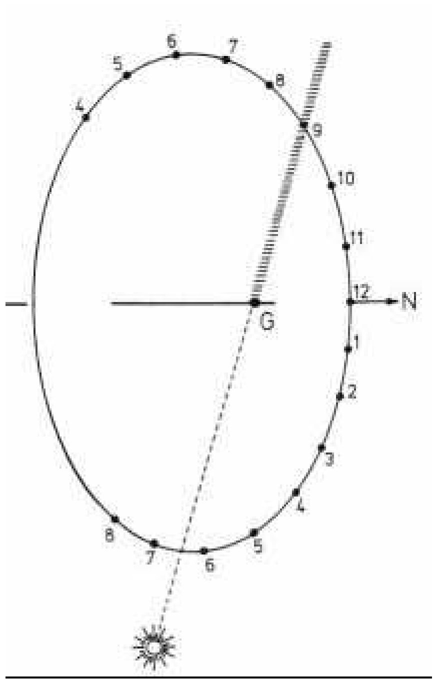


Figure 1. View from above of the prototype of an equator projection sundial. G is a vertical gnomon at a point on the line of dates. The shadow of the gnomon indicates the true solar time on the circumference of the ellipse (here 9 o'clock, May l).

A modern example of a kind first described by Foster *(ref. 2)* in 1654 is the equiangular sundial constructed by Gordon E. Taylor *(ref. 7)* at Herstmonceux Castle, East Sussex. *(Now at Cambridge)* This sundial was constructed in 1975 on the occasion of the third centenary of the Royal Greenwich Observatory. Here the hour points lie on a circle of stainless steel with a diameter of 3.2m (10 ft). This circle makes an angle of about 40 degrees with the horizontal, and the movable gnomon is vertical. Because the hour points are placed at equal distances around the circle, corrections for longitude, the equation of time and summer time can be carried out simply by turning the graduated scale so that this sundial can show standard time.

The equiangular sundial at the Royal Greenwich Observatory, Herstmonceux Castle. The sundial now is at Cambridge. (picture differs from the one in the original article)

Orthographic projection of the equator circle

Figure 2 shows the celestial sphere, with the horizon, the equator and the polar axis, for a latitude fi and Figure 3 the equator circle with part of the polar axis, 'taken out of' Figure 2. If the declination of the Sun is d, the part of the axis that appears above the equator is Rtan d (where R is the radius of the sphere). The top of this section of the axis is marked T. The shadow of T will fall exactly on the equator circle during the whole day, and it will move 15 degrees during each hour. This makes it is easy to construct hour points on the equator.

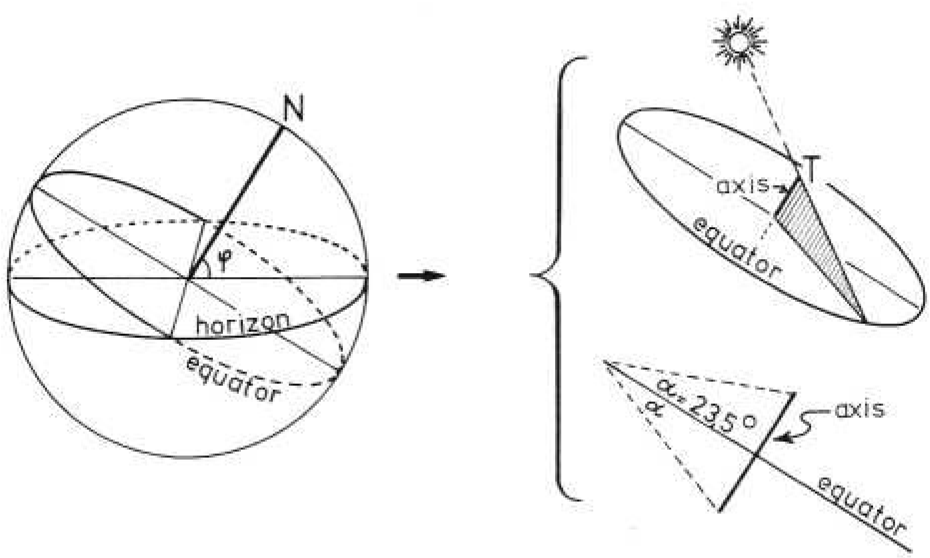


Figure 2.(left) The celestial sphere with horizon, equator and polar axis in perspective.
Figure 3. (right) Meridian section, taken from Figure 2.

In Figure 4 the equator circle has been drawn again with the same part of the polar axis, and here it is also projected onto a horizontal plane. The projection of the equator circle is an ellipse with the short axis in the N-S direction. A is an hour point (for example, the 11 o'clock point) on the equator circle. The projection of T and A are T' and A'; so A' is also the 11 o'clock point on the ellipse.

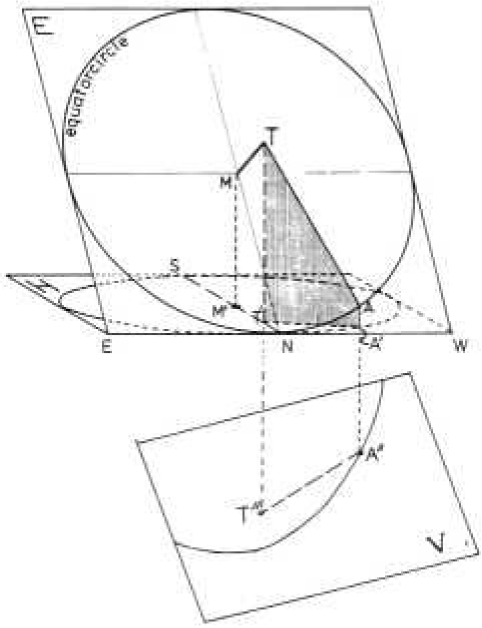


Figure 4. Perpendicular projection of the equator circle on a horizontal plane. The shadow plane from TT' intersects the hour point A as well as the projection of the hour point A'.

Now imagine that TT' is a gnomon for the horizontal dial. TT'AA' is then part of the shadow surface that the Sun casts of the gnomon. In the course of the day A, the shadow point of T, will travel along the equator circle with constant velocity, and at the same time the shadow line T'A' will intersect the ellipse in the hour points which are the rojections of the hour points on the equator circle. In the course of the year the gnomon TT' must be moved along the N-S line because T changes with the declination of the Sun. With this diagram we have constructed the prototype of the equator projection sundial: an ellipse with hour points plus a perpendicular gnomon which must be moved along the N-S line according to the value of the Sun's declination. So far, the derivation of this sundial is almost the same as Terpstra's. Now we give the following development of it. If in Figure 4 we turn the plane H round the N-S axis, or round the E-W axis, or round both, T will be projected on this plane V as T" and A as A", but T"A" is always a shadow line which shows the same hour on the projected equator circle as the shadow point A on the equator circle. So we may choose the projection plane at will.

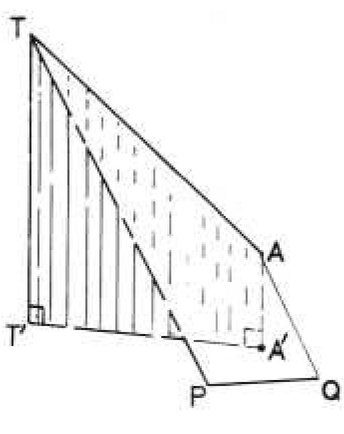


Figure 5. The direction of projection can be chosen at will.

The direction of the projection may also be chosen at will, as is shown in Figure 5. TAA'T' is the same plane as in Figure 4. We choose any point P of the projection plane and we fix with this a new direction of projection TP. When the hour point is projected in Q, where AQ is parallel to TP. If we use TP as gnomon, TAQP is the shadow plane and the shadow line PQ will indicate in Q the same time as TA in A. We can conclude that if we project the equator circle with the hour points plus part of the polar axis (with a length of 2Rtan 23.5) in any direction on any plane V, and if we put the gnomon on a place corresponding with the date on the projection of the polar axis parallel to the direction of the projection, the shadow of the gnomon indicates true solar time on the projected equator circle. This situation is shown again in Figure 6.

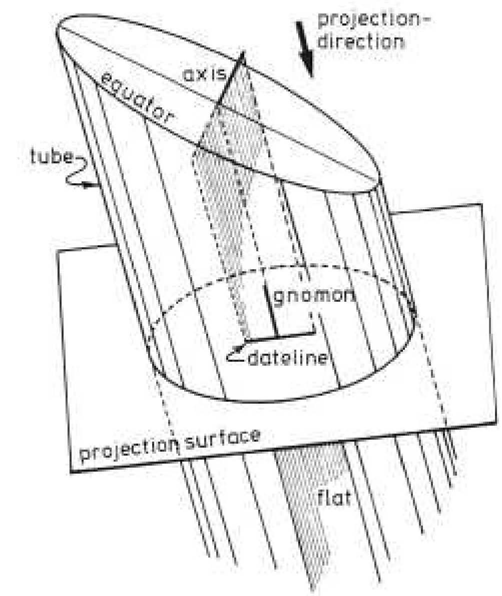


Figure 6. A sundial always results if the equator circle and axis are projected on any plane in any direction.

Because the lines of projection from the equator circle form a cylinder, of which the perpendicular section is an ellipse, any projecting plane will intersect this cylinder in general as an ellipse. There are however two circular sections: the equator circle itself and the mirror image of the equator circle in a plane at right angles to the axis of the cylinder. (We will return to this subject below.) With a certain choice of the projection direction, the projection of the circle degenerates into a line. We will describe all kinds of sundials which arise in this way as equator projection sundials. All the already known analemmatic sundials belong to this group. By this method they can be constructed simply, and from this construction the goniometric relations (desirable for a precise ealisation) can be deduced.

A few examples

To characterise the different directions of projection and the projection planes, we start from the projection of the equator circle and axis on the meridian plane. In the corresponding figures the projection lines have been dotted and the projection plane has been drawn as a double line.

a) A vertical sundial is possible, of course. The vertical plane is just one of the endless possibilities. After choosing vertical projection plane, the direction of projection is still completely free.

b) We get rectilinear sundials, where the hour points are on a straight line, if the direction of projection is parallel to the equator plane. Thus the number of possibilities is unlimited, and the choice of the projection plane is still free afterwards. Figure 7 shows a sundial for an east wall with a gnomon that is at right angles to that wall. The direction of projection (left side of the figure) is perpendicular to the drawing plane.

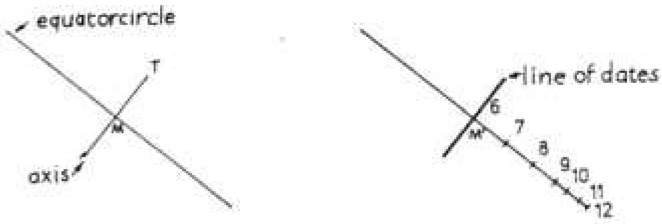


Figure 7. A linear sundial.

c) All sundials with a polar style can also be derived from equator projection. We simply choose the direction of the polar axis as the direction of projection, and the line of dates then shrinks to a point. The choice of the projectionplane is free.

d) When part of the line of dates is situated outside the ellipse, it is possible that on some dates the shadow of the gnomon can reverse its direction of rotation at certain times of day. *(ref. 8)* With reference to the Book of Kings, such a dial is called an Achaz sundial. We are not going further into this matter here, but in Figure 8 we give a direction of projection that fulfills the conditions necessary for a reversing shadow. There are many possibilities again.

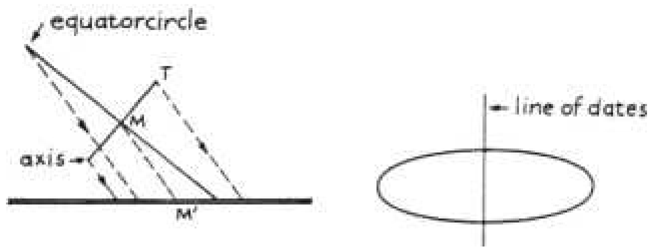


Figure 8. The line of dates falls partly outside the ellipse.

e) With a suitable choice of the direction of projection, it is possible to make the line of dates as long as the axis of the ellipse, with a free choice of the projection plane (Figure 9).

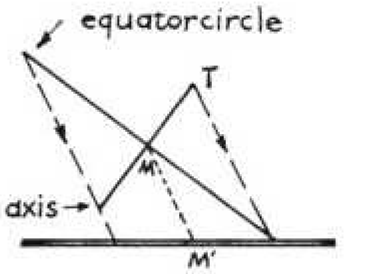


Figure 9. The line of dates is as long as the short axis of the ellipse.

f) It is not necessary that the short axis of the ellipse lie in the N-S direction. Figure 10 shows one of the possibilities for a sundial where the long axis lies in the N-S direction.

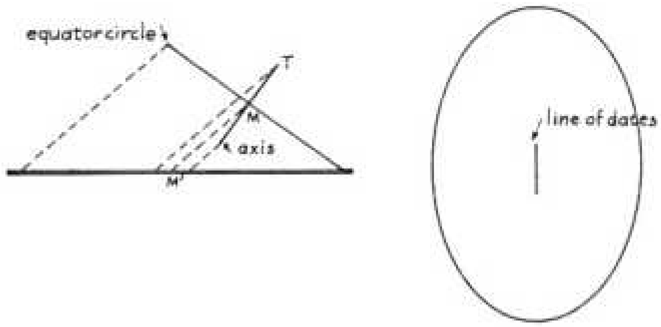
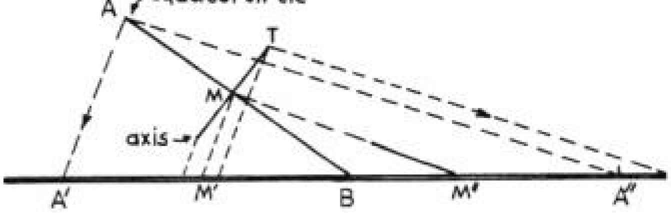


Figure 10. The long axis of the ellipse is directed N-S.

g) Homogeneous sundials are interesting. In such a dial, the hour points are situated on a circle at equal distances. We can get this situation in two ways. In Figure 11 we start from a given projection plane. We make A'B = AB. Then AA' is the direction of projection that produces a circular carrier for the hour line. There is also another possibility here. On the right hand side of B we choose A" so that A"B = AB. Then the direction of projection is AA". The second method is to start with a given direction of projection (Figure 12). The projecting line through A is l. Make BA' = BA whereby A' is stituated on l. Then A'D is the meridian section of the projecting plane.



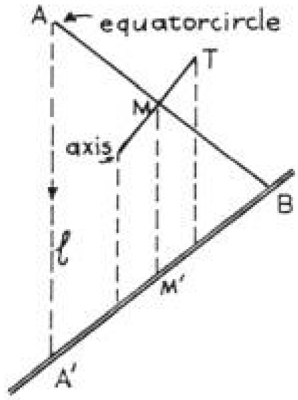


Figure 11. (left) Two possible homogeneous sundials on a given projection plane.
Figure 12. (right) A homogeneous sundial with given direction of projection.

h) It is also possible of course to use completely arbitrary directions of projection and projection planes. In general one would hardly consider constructing one of these. Only in special circumstances (for instance for a sundial on a surface that has been fixed by architectural considerations) would such a sundial be desirable.

A plural homogenous sundial

We want to describe one sundial in a bit more detail. In Figure 11 we see two sundials in which the hour lines are situated on circles with equal radii. We can put both circles on top of each other; then we have one circle with two gnomons. The two lines of dates are generally not of the same length. But if we choose a projection plane parallel to the polar axis, the lines of dates become equal. In Figure 13 this plane has been placed through OZ. By circling NM to the left and to the right (NZ and NO) two circular sundials arise. Here the lines of dates are equal and we can put both circles on top of each other. The double gnomon has a V-shape, with the two gnomons at right angles. This double sundial can be extended in a simple way with a linear one. In Figure 13 we choose the direction of projection MN. The equator becomes a line and the line of dates as long again as TS. The third gnomon is now perpendicular to the projection plane.

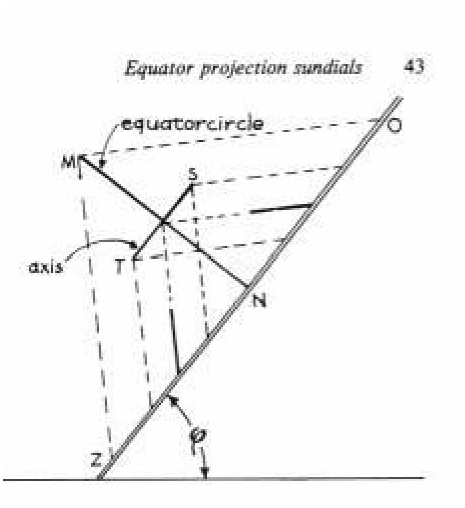


Figure 13. Direction of projection and projection plane chosen so that two homogeneous sundials arise with equal lines of dates but differently directed gnomons.

In a model of this triple sundial (Figure 14) I used a plastic drawing triangle as gnomon (ABV). Gnomon AV casts a shadow on the upper half of the circle and BV on the lower half. Gnomon VC belongs to the linear sundial EW. The point line of dates is PQ. Because the projection plane runs parallel to the polar axis, AB can also be used as the gnomon for a polar sundial. The hour lines of this are circled in the figure. (This quadruple sundial is self directing of course).

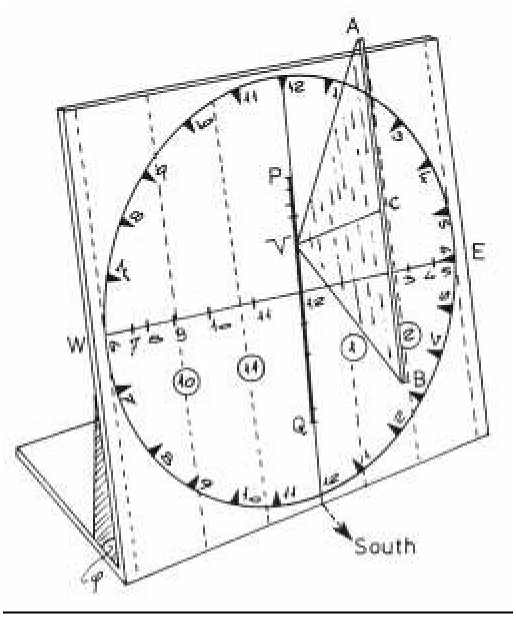


Figure 14. Quadruple sundial: two have the same hour point circle. The third one is linear and the fourth is a normal polar sundial.

Central projection

It is also possible to project the equator circle and axis from a point, and I wondered if it would be possible to construct sundials in this way.

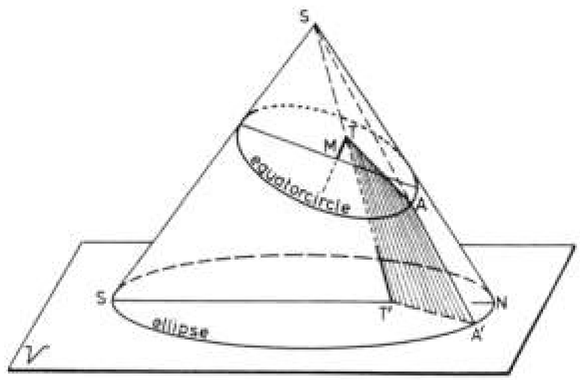


Figure 15. A sundial also arises with projection from a point.

In Figure 15 the equator circle has been projected from S onto V. A gnomon STT' appears to cast on V a shadow line T'A', which intersects the ellipse in a point A' that is the projection of the hour point A of the equator circle. Thus the ellipse in plane V with hour points projected on it forms a sundial with the gnomon TT'. The projection plane and projection centre can be chosen at will here too. The proof of this is the same as we have given for the orthographic projection. This projection results in a noteworthy group of sundials, which is new as far as I know.

They have the following properties:

The projection plane always intersects the cone as a conical section. So now we can consider not only the ellipse, circle and line as the bearer of the hour point, but also the parabola and hyperbola.

The gnomon must not only be moved according to the date, but at the same time we must change its direction to ensure that the gnomon always points to the projection centre S. This second requirement seems difficult to achieve in practice. However, by describing a sundial from this group we will show how the simultaneous move and change in direction can be realised in a simple way.

Example of a central projection sundial

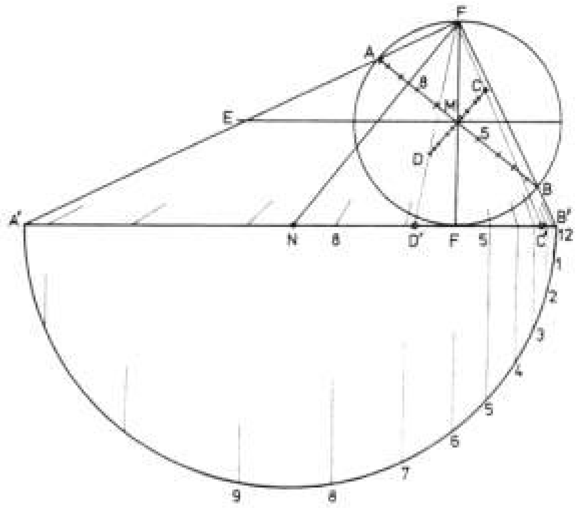


Figure 16. The construction of a sundial by means of central projection, with the hour points situated on a circle.

In Figure 16 the meridian section AB of the equator has been drawn for latitude fi (here 52 degrees). The hour points have been constructed on AB as we see them projected on the meridian plane. A scale of dates has been constructed on axis DC. The projection centre F is chosen so that the equator circle will be projected on the horizontal plane as a circle. We achieve this by drawing a circle with diameter AB. FMF' is a perpendicular line on the horizontal plane. If we imagine the circle to be a picture of the sphere then we have here a stereographic projection of the equator circle from F on the plane perpendicular to FF'. With the stereographic projection a circle on a sphere is always projected as a circle. So A'B' is the diameter of the projected equator circle. Half of this circle has been drawn as it looks on the horizontal plane. We project the hour points from AB onto A'B', and then we move them onto the circumference of the half circle. Then the date points are projected onto A'B'. (In Figure 16 only the end points D' and C' of the date line have been projected.) With this we have finished the construction. Above the hour line plane we must put a fixed point F and it must be possible to direct the gnomon from any point of the line of dates towards F. There is an interesting solution to this problem. We combine this equator projection sundial with a horizontal sundial with polar style. One point of this style can serve to fasten a piece of cord which forms the movable gnomon for the equator projection sundial. In Figure 16 it appears that NF is perpendicular to AB (this is simple to prove). Hence NF can serve as polar style for a horizontal sundial.

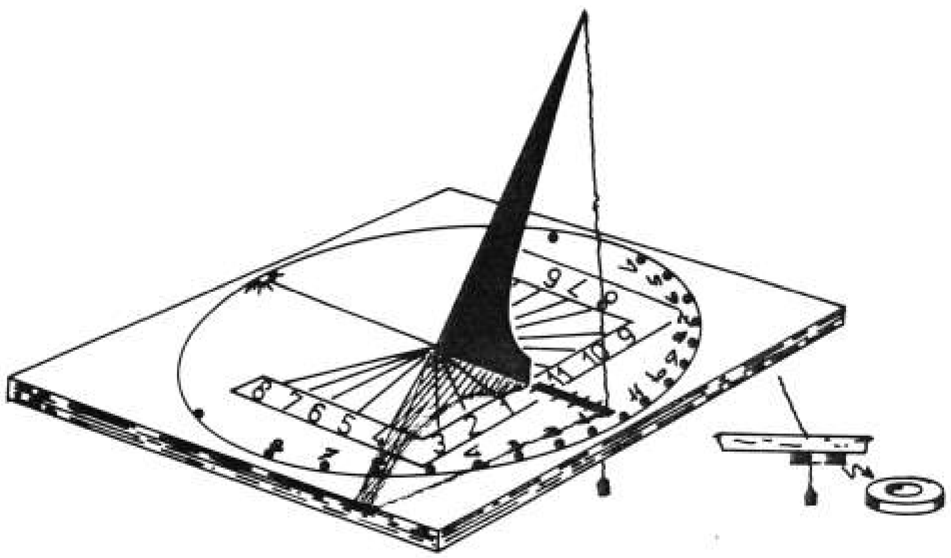


Figure 17. The practical construction of the sundial of Figure 16 is combined with a polar style.

The result is shown in Figure 17. The polar style (length NF) stands in the centre. The matching sundial with hour lines is circumscribed by a rectangle. The hour points for the equator projection sundial are on a circle; the matching line of dates can be seen as a N-S directed groove. The gnomon of this sundial is formed by a piece of cord that goes from the top of the polar style to the relevant point on the line of dates. The cord is anchored in the required position by a ring-shaped ferrite magnet, which attaches itself to a flat piece of iron that is affixed below the line of dates. The double sundial is self directing because the indication of time is based on two different principles *(ref. 9)* '. The polar style is in the meridian plane only when both sundials indicate the same time (and when they indicate 12 o'clock as is the case with most self directing sundials).

Remarks

The name 'analemmatic sundial' has not been very happily chosen. De Vaulezard *(ref. 1)* probably chose it because he used a projection of the sphere on the meridian plane for his construction. Since the time of Vitruvius (25 ac) this projection has been named the analemma. A little less then two centuries later, Ptolemy wrote an essay about the analemma; this is however a fairly complicated graphic method for the solution of problems in spherical trigonometry *(ref. 10)* , and the name is quite irrelevant in connection with this type of sundial. Nor is the term 'azimuthal sundial', used by Foster *(ref. 2)* (and after him by many others) suitable to describe the whole family of sundials. Only with a horizontal projection plane and a vertical gnomon does the shadow of the gnomon indicate the Sun's azimuth. I suggest they should be called 'equator projection sundials' because this name describes a principle of construction with which all already known analemmatic sundials and a number of new forms can be produced.

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